Small-scale perturbations

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In previous lectures we considered the homogeneous and isotropic Universe. But small irregularities must exists. These irregularities created large scale structure of our Universe, they formed "pancakes" of matter with high density contrast in which cluster of galaxies, galaxies , stars and planets were created.

In the following lectures it will be assumed that the size of density perturbations are much less than ct where t being the physical time of our Universe. It will be assumed also that the motion of the matter is nonrelativistic, and peculiar velocity $\boldsymbol{v}$ everywhere much less then the speed of light.

Perturbations of the fluid in static background matter

First of all I would like to consider a sound wave in ordinary fluid with self gravitation. We have three dynamical equations:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{u}) = 0 \qquad \rho(\vec{r}, t), \quad u(\vec{r}, t)$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\vec{\nabla})\vec{u} = -\vec{\nabla}\Phi - \frac{1}{\rho}\vec{\nabla}p \qquad p(\vec{r}, t)$$

$$\Delta\Phi = 4\pi G\rho \qquad \Phi(\vec{r}, t)$$

One additional equation is necessary. It is equation of state $p=p(\rho)$
This equation is nonrelativistic equation of state, and we will compare p Over $\rho\, v_s^2$, where v_s is ordinary speed of sound.

A sound wave is small perturbation of background fluid which has constant background density, pressure, zero background velocity and constant background potential Φ.

Therefore, we can write perturbations as

$$\rho(\vec{r},t) = \rho_0(1+\delta(\vec{r},t))$$

$$\vec{u}(\vec{r},t) = 0+\vec{v}(\vec{r},t)$$

$$p(\vec{r},t) = p_0 + \delta p(\vec{r},t)$$

and $$\Phi(\vec{r},t) = \Phi_0 + \varphi(\vec{r},t)$$

We must find the δp using equation of state. To do it I expand general equation of state in Taylor series around ρ_0 : $p_0 = p(\rho_0)$

$$p(\rho) = p_0 + \frac{1}{1!}\frac{\partial p}{\partial \rho}(\rho - \rho_0) + \frac{1}{2!}\frac{\partial^2 p}{\partial \rho^2}(\rho - \rho_0)^2 + ...$$

$$\frac{\partial p}{\partial \rho} = v_s^2 \quad \text{and} \quad p(\rho) = p_0 + v_s^2 \rho_0 \delta \ \Rightarrow$$

$$\delta p = v_s^2 \rho_0 \delta$$

Substituting this result into dynamical equation we obtain the system of equations

$$\rho_0 \frac{\partial \delta}{\partial t} + \rho_0 (\vec{\nabla} \vec{v}) = 0$$

$$\frac{\partial \vec{v}}{\partial t} = -\vec{\nabla} \varphi - v_s^2 \vec{\nabla} \delta$$

$$\Delta \varphi = 4\pi G \rho_0 \delta$$

I would like to consider the solution of these equations in the case of monochromatic plane sound wave.

So, we will choose the particular solutions of partial differential equations

$$\delta(\vec{r},t) = \delta_0(t)\exp(i\vec{k}\vec{r})$$

$$v(\vec{r},t) = v_0(t)\exp(i\vec{k}\vec{r})$$

$$\varphi(\vec{r},t) = \varphi_0(t)\exp(i\vec{k}\vec{r})$$

with unknown functions of time

$$\delta_0(t) \qquad v_0(t) \qquad \varphi_0(t)$$

If we substitute these solutions into above partial differential equation we two obtain ordinary differential equations and one algebraic equation

$$\frac{d\delta_0}{dt} + i(\vec{k}\vec{v}_0) = 0$$

$$\frac{d\vec{v}_0}{dt} = -i\vec{k}\varphi - i\vec{k}v_s^2\delta_0$$

$$-k^2\varphi = 4\pi G\rho_0\delta_0$$

the first step of solution is substitution of algebraic equation

$$\varphi_0 = -4\pi G\frac{\rho_0\delta_0}{k^2}$$

into second differential equation. The second step is multiplication of second equation by ***k*** and we obtain the new system

$$\frac{d\delta_0}{dt} + i(\vec{k}\vec{v}_0) = 0$$

$$\frac{d(\vec{k}\vec{v}_0)}{dt} = i(4\pi G\rho_0 - k^2 v_s^2)\delta_0$$

or $\Rightarrow$ $$\frac{d^2\delta_0}{dt^2} - (4\pi G\rho_0 - k^2 v_s^2)\delta_0 = 0$$

I would like to introduce definition

$$\Omega^2 = 4\pi G\rho_0 - k^2 v_s^2$$

and the solution of the last equation is

$$\delta_0(t) = \delta_1 \exp(\Omega t) + \delta_2 \exp(-\Omega t)$$

increasing mode $\uparrow$ $\uparrow$ decreasing mode

The value Ω is real for $k \to 0$

and it is imaginary if $k \to \infty$

There exists the critical wavelength

$$\lambda_J = \sqrt{\frac{\pi v_s^2}{G\rho_0}}$$

IT IS JEANS LENGTH

If the length of a wave is larger then Jeans wavelength of our fluid

$$\lambda > \lambda_J$$

then $\Omega^2 > 0$

and we have superposition of increasing and decreasing modes.

In opposite case, if $\lambda < \lambda_J$

we obtain that is $\Omega^2 < 0$ and the solution is

$$\delta_0(t) = \delta_1 \exp(i\,|\,\Omega\,|\,t) + \delta_2 \exp(-i\,|\,\Omega\,|\,t)$$

which describes usual sound wave.

Therefore, we can expect the increasing of density perturbation due to gravitational instability in the fluid with long waves perturbations, and we can expect ordinary sound waves in opposite case

$$\lambda < \lambda_J$$

Now, we have to consider the evolution of density perturbation due to gravitational instability in the expanding Universe

Perturbations of the fluid in the expanding Universe

The qualitative conclusion is valid also in the expanding Universe, but numerical values are different.

What are main differences in the case of expanding Universe?

The equation of motion written in eulerian coordinates are the same, but the decomposition into background values and small perturbations is other

They are

$$\rho(\vec{r},t) = \rho(t)(1+\delta(\vec{r},t))$$

$$\vec{u}(\vec{r},t) = \frac{\dot{a}}{a}\vec{r} + \vec{v}(\vec{r},t)$$

$$\Phi(\vec{r},t) = \frac{2\pi}{3}G\rho r^2 + \varphi(\vec{r},t)$$

Now the equation of mass conservation is

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{u}) = 0 \quad \Rightarrow$$

$$\frac{d\rho}{dt} + 3\frac{\dot{a}}{a}\rho = 0$$

$$\frac{\partial\delta}{\partial t} + \frac{\dot{a}}{a}(\vec{r}\vec{\nabla})\delta + (\vec{\nabla}\vec{v}) = 0$$

So, we have unperturbed equation and first order perturbation one. It is convenient to change spatial variables from eulerian coordinates ***r***, to comoving lagrangian. Coordinates in unperturbed model, defined by equation

$$r^{\alpha} = a(t)x^{\alpha}$$

Then according to usual rule of differetiation we transform the equation via

$$\frac{\partial}{\partial t}\Big|_x = \frac{\partial}{\partial t}\Big|_r + \frac{\dot{a}}{a} r^{\alpha} \frac{\partial}{\partial r^{\alpha}}\Big|_t$$

and the equation for perturbation becomes

$$\frac{\partial \delta}{\partial t}\Big|_x + \frac{(\vec{\nabla}\vec{v})}{a} = 0$$

The same procedure is applied to the second dynamical equation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\vec{\nabla})\vec{u} = \frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}(\vec{r}\vec{\nabla}\,|_r)\vec{v} + \frac{\dot{a}}{a}\vec{v} \quad \Rightarrow$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v}$$

so, we obtain

$$\frac{\partial \delta}{\partial t} + \frac{(\vec{\nabla}\vec{v})}{a} = 0$$

$$\frac{\partial \vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} = -\frac{\vec{\nabla}\varphi}{a} + \frac{v_s^2 \vec{\nabla}\delta}{a^2}$$

$$\Delta\varphi = 4\pi G\rho(t)\delta(t,\vec{r})$$

Let us consider a simple example. Suppose that we have pressureless gas, the pressure is equal to zero and velocity of sound is also equal to zero.

In this case we have

$$\frac{\partial\delta}{\partial t} + \frac{(\vec{\nabla}\vec{v})}{a} = 0$$

$$\frac{\partial\vec{v}}{\partial t} + \frac{\dot{a}}{a}\vec{v} = -\frac{\vec{\nabla}\varphi}{a}$$

$$\Delta\varphi = 4\pi G\rho(t)\delta(t,\vec{r})$$

If we transform these equations as previously we obtain the equation for density contrast

$$\frac{d^2\delta}{dt^2} + \frac{4}{3t}\frac{d\delta}{dt} - \frac{2}{3t^2}\delta = 0$$

Let us search the solution in the form

$$\delta(t) = \delta_0 t^{\alpha}$$

In this case the differential equation transform into algebraic one

$$\alpha(\alpha - 1) + \frac{4}{3}\alpha - \frac{2}{3} = 0 \quad \text{or} \quad \alpha = -\frac{1}{6} \pm \sqrt{\frac{25}{36}} = \frac{2}{3}, \quad -1$$

So, the solution is

$$\delta(t) = \delta_1 t^{2/3} + \delta_2 t^{-1}$$

growing mode — decreasing mode

Let us consider now the ordinary gas with pressure, velocity of sound is not equal to zero now

The first term in the solution is growing mode, the second is decaying mode. We are interested in growing mode only.

So, one can say, that in the case of the expanding Universe Jeans instability changes from exponential to power law time dependence.

The mass of a clump of matter with size equals the Jeans length is called the Jeans mass. It estimates the minimum mass of an object formed due to the Jeans instability.

$$M_J = \rho_m \lambda_J^3$$

Let us estimate the Jeans mass just after the recombination epoch in the Universe filled with baryons.

Here we have to substitute ρ_B

Instead of matter density and obtain

$$M_J = \frac{\pi^{3/2}}{G^{3/2}} \frac{v_s^3}{\sqrt{\rho_B}} \approx 10^6 M_{sun}$$

It is necessary to mention here that our Universe filled with dark matter. In the LCDM (standard model), it is the collisionless gas of particles with very low effective temperature. Therefore, the effective pressure of dark matter is negligibly small, and Jeans instability develops in dark matter much before recombination, just after the epoch of transfer from radiation dominated equation of state to a matter dominated equation of state in the early Universe.

Let us consider now the ordinary gas with pressure, velocity of sound is not equal to zero now

$$v_s \neq 0$$

The general equation is now

$$\frac{d^2\delta}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\delta}{dt} + \left(\frac{k^2 v_s^2}{a^2} - 4\pi G\rho(t)\right)\delta = 0$$

$$k = \frac{2\pi a}{\lambda}$$

λ is physical wavelength

$k = \mathrm{const}$ during expansion

In the case of ideal gas

$$v_s^2 = \frac{\bar{k}T}{m_p} \quad \text{and} \quad \frac{k^2 v_s^2}{a^2} \propto \frac{1}{a^3}$$

In this case the solution of density equation is similar to the pressureless equation. In the case of dust-dominated background, flat Universe and ideal gas we can rewrite the term in parenthesis as

$$\left(\frac{k^2 v_s^2}{a^2} - 4\pi G\rho(t)\right) = \left((\frac{2\pi}{\lambda_0})^2 \frac{\bar{k}T}{m_p} \frac{t_0^{2/3}}{t^2} - \frac{4\pi G\rho_0 t_0^2}{t^2} \right) =$$

$$4\pi G\rho_0 t_0^2 \left(\frac{\lambda_J^2}{\lambda_0^2} - 1\right)\frac{1}{t^2} = \omega_0^2(\mathrm{t}) \quad \text{where} \quad \lambda_J^2 = \frac{\pi \bar{k} T_0}{G\rho_0 t_0^{4/3}}$$

As a result we obtain the equation for density contrast

$$\frac{d^2\delta}{dt^2} + \frac{4}{3t}\frac{d\delta}{dt} + 4\pi G\rho_0 t_0^2\left(\frac{\lambda_J^2}{\lambda_0^2} - 1\right)\frac{1}{t^2}\delta = 0$$

Now we can choose the solution as

$$\delta(t) = \delta_0 t^\alpha \qquad \text{where}$$

$$\alpha(\alpha - 1) + \frac{4}{3}\alpha + 4\pi G\rho_0 t_0^2\left(\frac{\lambda_J^2}{\lambda_0^2} - 1\right) = 0$$

and the solution is

$$\alpha = -\frac{1}{6} \pm \sqrt{\frac{1}{36} + 4\pi G\rho_0 (1 - \frac{\lambda_J^2}{\lambda_0^2})}$$

$$\text{if} \quad \frac{1}{36} + 4\pi G\rho_0 (1 - \frac{\lambda_J^2}{\lambda_0^2}) > 0$$

We have superposition of increasing and decreasing modes

In opposite case, if

$$\frac{1}{36} + 4\pi G\rho_0 (1 - \frac{\lambda_J^2}{\lambda_0^2}) < 0$$

we have $\alpha = -\frac{1}{6} \pm i\omega$

and the general solution is

$$\delta(t) = \delta_1 t^{-1/6} \mathrm{e}^{i\omega \ln t} + \delta_2 t^{-1/6} \mathrm{e}^{-i\omega \ln t}$$

So, we have oscillating wave with monotonically decreasing amplitude.

We considered the case of ideal gas. What is the main difference in the case of hot plasma?

Radiation drag. Silk effect.

In previous material I demonstrated that due to gravitational instability small-scale perturbations can grow and can significantly increase density contrast in the almost homogeneous and isotropic Universe filled with ideal fluid.

What is main difference in the case of the hot Universe or Big Bang?

As long as the matter is neutral the only effect of the Primeval Fireball radiation is to increase somewhat the rate of expansion. For instance, in radiation dominated model

$$p = \rho c^2 / 3$$

We have the Hubble parameter as function of time

$$\dot{a}/a = (2t)^{-1}$$

and the solution for density perturbation is

$$\delta(t) = \delta_1 t^{\sqrt{3/8}} + \delta_2 t^{-\sqrt{3/8}}$$

It is not very different from

$$\boldsymbol{\delta(t) = \delta_1 t^{2/3} + \delta_2 t^{-1}}$$

However, if the protons and electrons of primordial plasma is coupled to the radiation it can very significantly alter the picture.

Consider an electron with peculiar velocity much smaller then the speed of light

$$v << c$$

In the electron frame of reference the distribution of radiation temperature is

$$T(\theta) = T(1 + \frac{v}{c}\cos\theta)$$

and the radiation brightness is

$$B(\theta) = c\sigma T^4(\theta)/4\pi$$

Due to effect of Thomson scattering the force of radiation pressure emerges

$$F = -\int \sigma_T \frac{B(\theta)}{c} \cos\theta d\Omega$$

which is

$$F = -\frac{4\sigma_T}{3} \sigma T^4 \frac{v}{c}$$

It is the radiation drag force on the electron

The radiation acts on electrons and protons of primordial plasma. However, due to the fact that the Thomson cross section of electrons are much larger than the cross-section of protons we have that radiation pressure on electrons is larger. The electrostatic forces, which emerge in the plasma if the electrons and protons are separated, keep the stability of plasma and reduces separation to zero keep the plasma electrically neutral.

The radiation drag force per unit volume in opposite direction of the plasma motion is

$$F = -\frac{4\sigma_T}{3}\sigma T^4 \frac{v}{c}\frac{\rho}{m_p}$$

Where

$$\frac{\rho}{m_p}$$

is density of electrons per unit volume.

When this force is added to equation of density perturbation, it becomes

$$\frac{d^2\delta}{dt^2}+\left(2\frac{\dot{a}}{a}+\frac{4}{3}\sigma_T\frac{\sigma T^4}{m_p c}\right)\frac{d\delta}{dt}+\left(\frac{k^2}{a^2}\frac{\bar{k}T}{m_p}-4\pi G\rho(t)\right)\delta=0$$

Let us estimate the ratio of two terms in first parenthes is

$$\frac{2\sigma_T \sigma T^4}{3Hm_p c} \approx 10^{-6} h^{-1} (1+z)^{5/2}$$

When $z > 1000$ the ratio is larger then 30 which means that radiation drag dominates the left hand side of equation during radiation-dominated era. Therefore, we can leave only one term in the left hand side of this equation. If we also neglect the pressure term we obtain

$$\left(\frac{4}{3} \sigma_T \frac{\sigma T^4}{m_p c} \right) \frac{d\delta}{dt} = 4\pi G \rho(t) \delta$$

The solution is

$$\delta(t) = \delta_0 \exp\{\int_0^t \frac{3\pi G\rho m_p c}{\sigma_T \sigma T^4} dt\} = \delta_0 \exp\{\frac{3 \bullet 10^5}{(1+z)^{5/2}}\}$$

It gives $\frac{\delta(t)}{\delta(0)} = 1.006$ at $T = 3000^o \quad K$

We can write the equation which describes the evolution of density perturbations in the radiation dominated plasma having arbitrary wavelength

$$\left(\frac{4}{3}\sigma_T \frac{\sigma T^4}{m_p c}\right)\frac{d\delta}{dt} + \left(\frac{k^2 v^2}{a^2} - 4\pi G\rho(t)\right)\delta = 0$$

The term in parenthesis has negative sign if wavelength is larger then Jeans wavelength and positive sign in opposite case. It means that amplitude of a wave with wavelength larger then Jeans one remains almost constant during the radiation-dominated epoch with very small increase. The amplitude of a wave having $\lambda < \lambda_J$ decreases exponentially.

The term σT^4 is energy density of photon radiation.

Total energy density is proportional to T^4 and the number of degrees of freedom of all species of relativistic particles.

So, we can transform both

$$\sigma_T \frac{\sigma T^4}{m_p c}$$

and

$$4\pi G\rho(t) - \frac{k^2 v_s^2}{a^2}$$

to the convenient form. It is convenient to introduce the characteristic time

$$\tau = \frac{16}{9N_g}\alpha^2 \frac{m_{pl}^2}{m_p m_e}\frac{\lambda_c}{c} = 3 \bullet 10^{16}\ s$$

Here N_g is the number of degrees of freedom, α is the fine structure constant, m_p is proton mass, m_e is electron mass, $m_{pl} = 2\ 10^{19}$ Gev is Planck mass and λ_c is compton wavelength.

We can rewrite the equation as

$$\tau \frac{d\delta}{dt} = \left[1 - \frac{3\pi^2}{8}\left(\frac{l_H}{\lambda}\right)^2\left(\frac{v_s}{c}\right)^2\right]\delta$$

So, even for $\lambda >> \lambda_J$

That means $\left(\frac{l_H}{\lambda}\right)^2 \left(\frac{v_s}{c}\right)^2 << 1$

To the end of recombination one has

$$\delta(t) = \delta_0 \exp\left({}^{t_r}\!/_{\tau}\right) = \delta_0 \exp(6 \bullet 10^{-3})$$

In the case of short wavelength the term in parenthesis becomes negative and we have rapid decay in amplitude.

The initial density perturbations are decomposed into Fourier series.

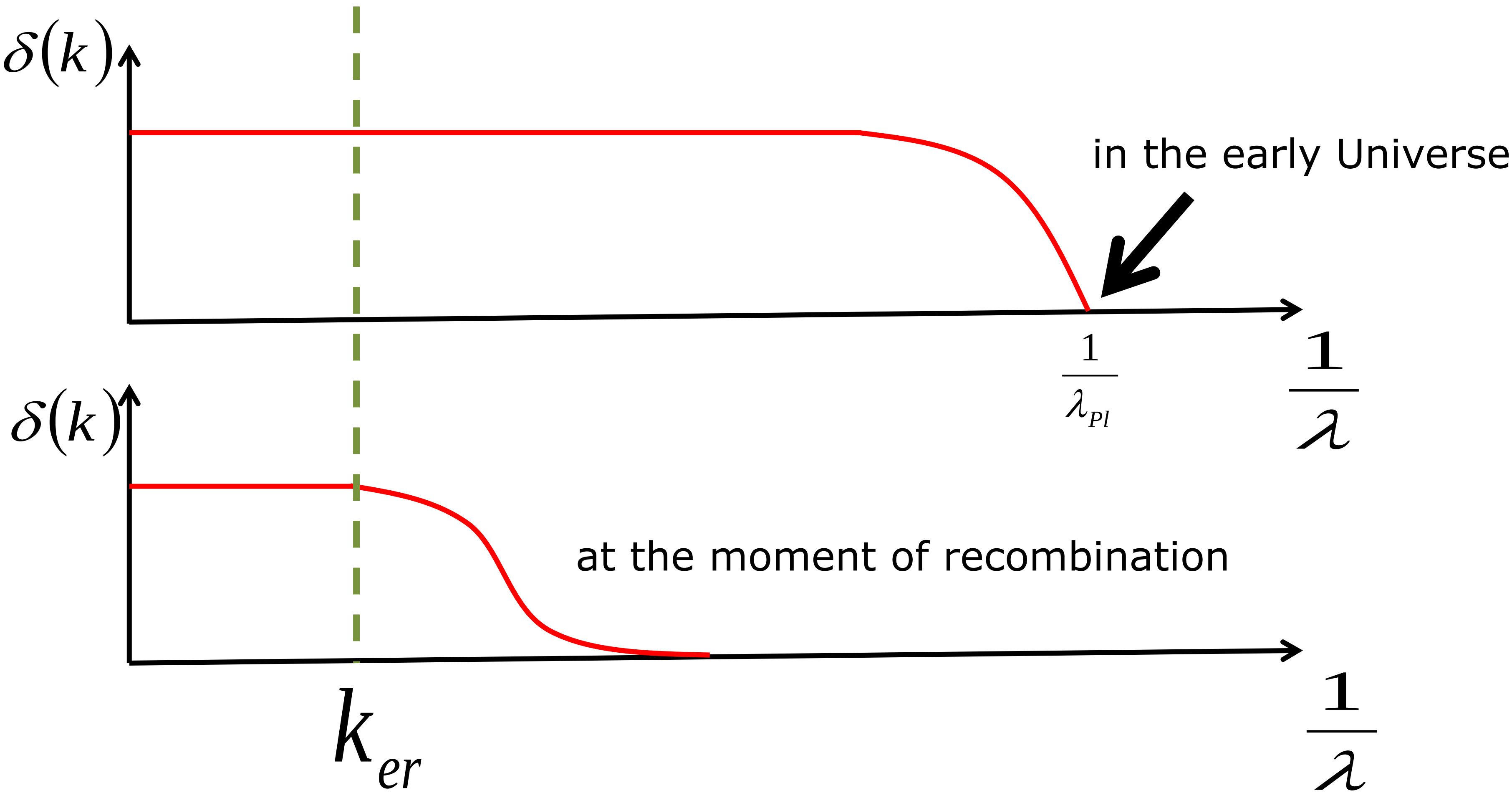
$$\delta(t,\vec{r}) = \int d^3k e^{i\vec{k}\vec{r}} \delta(t,\vec{k}) = \int d^3k e^{i\vec{k}\vec{r}} \delta(t,\frac{\vec{n}}{\lambda})$$

For the Fourier component $\delta(t,\frac{\vec{n}}{\lambda})$

The above mentioned equation is valid

$$\tau \frac{d\delta_\lambda}{dt} = \left[1 - \frac{3\pi^2}{8}\left(\frac{l_H}{\lambda}\right)^2\left(\frac{v_s}{c}\right)^2\right]\delta_\lambda$$

So each wave evolves in according to its own rule. It is necessary to remember that the second term in parenthesis is a function of time, therefore condition changes and waves of many lengths vanished during the Universe expansion.



In general, a sound wave in imperfect fluid will be damped at a rate

$$\Gamma = \frac{k^2 \sigma T^4}{6n\sigma_T (nm_p + \frac{4}{3}\sigma T^4)} \left\{ \frac{160}{135} + \frac{nm_p}{\sigma T^4} \frac{nm_p}{nm_p + \frac{4}{3}\sigma T^4} \right\}$$

So the second wave will be damped in amplitude by a factor

$$D = \exp\left\{-\int \Gamma(t)dt\right\} = \exp\left\{-\left(\frac{M_c}{M}\right)^{2/3}\right\}$$

where M_c is a critical mass which is determined by

$$\int \Gamma(t)dt$$

and M is the mass within the length $\lambda/2$ of the sound wave. One can estimate the critical mass in the accepted cosmological model.

$$M_c = 8\times 10^{13}\ M_{sun} \frac{100\, km\, s^{-1} Mpc^{-1}}{H_0}$$

Sufficient damping may be expected, if

$$\left(\frac{M_c}{M}\right)^{2/3} \approx 10$$

One can conclude that it is around mass of a gigantic galactic.

The perturbations having long wavelength

$$\lambda >> \lambda_{cr}$$

Survive to the moment of recombination.

After this moment they grow in according to

$$\delta_k(t) = \left(\frac{t}{t_r}\right)^{2/3} \delta_k(t_r) = (1+z_r)\delta_k(t_r)$$

The redshift at the recombination epoch is 10^3 therefore we can expect the initial perturbation to grow by a factor of 10^3.

We observe some quasars at $z=8$ and a lot of galaxies between $z = 0$ and $z=1$. Therefore one can expect well developed density contrast at $z = 0$. To have density contrast $\delta = 1$ at present moment we must have $\delta_r = 10^{-3}$ at recombination. But the observation of angular anisotropy of the CMBR shows that $\delta_r < 10^{-4}$.

This paradox was one of the reason to introduce the concept of invisible (dark, hidden) matter. Other reason came from galactic astronomy. The observation rotation curves of galaxies shows that the total (gravitational) mass of galaxies is greater than its visible mass.

Now the paradox is generally accepted that the matter of our Universe predominantly consists of dark matter. One part is called now just dark matter while the other is called dark energy. Here we will discuss the contribution of dark matter into evolution of density perturbations only.

The contribution of ordinary matter (baryonic matter) is of the order of 0.04 of total density of the Universe and the contribution of dark matter is of the order of 0.21, and the contribution of dark energy is of the order of 0.75 of total density of the Universe.

Now we must consider the evolution of density perturbation in the more complicated picture. So, we have two component matter which consists from baryonic matter having background density

$$\rho_b \approx 0.04\rho_{crit}$$

and from dark matter having background density

$$\rho_{DM} = 0.21\rho_{crit}$$

The total density of matter in the universe in our approach is equal to a fraction of the critical density

$$\rho_b + \rho_{DM} = 0.25\rho_{crit} = 0.25\frac{3H^2}{8\pi G}$$

Now we can expand the exact values of density, velocity and pressure to the first order approximation separately.

For instance,

$$\rho(t,\vec{r}) = \rho_{crit}\left(1+\Omega_b\delta_b + \Omega_{DM}\delta_{DM}\right)$$

here $\Omega_b = \dfrac{\rho_b}{\rho_{crit}}$ and $\Omega_{\mathrm{DM}} = \dfrac{\rho_{DM}}{\rho_{crit}}$

are density fractions of baryonic and dark matter

$\delta_{\mathrm{b}} = \dfrac{\delta\rho_b}{\rho_b}$ $\delta_{DM} = \dfrac{\delta\rho_{DM}}{\rho_{DM}}$ are density contrasts

After some transformations we obtain the coupled system of differential equations

$$\frac{d^2\delta_b}{dt^2}+\left(\frac{4}{3t}+\frac{3\tau}{8t^2}\right)\frac{d\delta_b}{dt}+\left(\frac{k^2 v_s^2}{a^2(t)}-4\pi G\rho_b\right)\delta_b=4\pi G\rho_{crit}(\Omega_{DM}\delta_{DM}+\Omega_b\delta_b)$$

and

$$\frac{d^2\delta_{DM}}{dt^2}+\frac{4}{3t}\frac{d\delta_{DM}}{dt}=4\pi G\rho_{crit}(\Omega_{DM}\delta_{DM}+\Omega_b\delta_b)$$

here Ω_{DM} and Ω_b are fractions of dark and baryonic matter and ρ_{crit} is time dependent critical density.

One can mention that $\Omega_{DM}\approx 5\Omega_b$ and density contrast in dark matter is larger than density contrast in baryonic matter. Therefore the term is right hand side of equations can be approximately written as

$$4\pi G\rho_{crit}\Omega_{DM}\delta_{DM}$$

Dark matter became nonrelativistic and obey the equation ($p=0$) after so-called decoupling epoch $z_d=10^5$. Until recombination

$$\delta_B \approx \mathrm{const}$$

or decreased very rapidly. At $z_d=10^5$ we have equal density contrast both in dark matter and in baryonic matter

$$\delta_b = \delta_{\mathrm{DM}}$$

But to the moment of recombination the perturbations of the dark component became greater and reached

$$\delta_{DM}(t_r) \approx \left(\frac{1+z_d}{1+z_r}\right)\delta_{DM}(t_d) \approx 100\delta_{DM}(t_d)$$

So, we have at the moment of recombination

$$\delta_D(t_r) \approx 100\delta_D(t_d)$$

Because of the fact that the perturbations of the baryonic component remain roughly constant

$$\delta_b(t_r) \approx \delta_b(t_d)$$

We will consider only long waves and do not consider back-action of the baryonic perturbations on the dark matter perturbations. At the moment of recombination the perturbations of dark matter are

$$\delta_{DM}(t_r) = \delta_{DM}\left(\frac{t_r}{t_d}\right)^{2/3}$$

and they continue to grow as

$$\delta_{DM}(t) = \delta_{DM}\left(\frac{t}{t_d}\right)^{2/3}$$

One can write the equation for baryonic component ($\lambda >> \lambda_J$) as

$$\frac{d^2\delta_b}{dt^2} + \frac{4}{3t}\frac{d\delta_b}{dt} - \frac{2}{3t^2}\Omega_b\delta_b = \frac{2}{3}\frac{\delta_{\mathrm{DM}}}{t_d^{2/3}} t^{2/3}\frac{1}{t^2}$$

Third term in left hand side is negligible small, therefore we can transform this equation into

$$\frac{d^2\delta_b}{dt^2} + \frac{4}{3t}\frac{d\delta_b}{dt} = \frac{2}{3}\frac{\delta_{\mathrm{DM}}}{t^{4/3} t_d^{2/3}}$$

One can transform it as

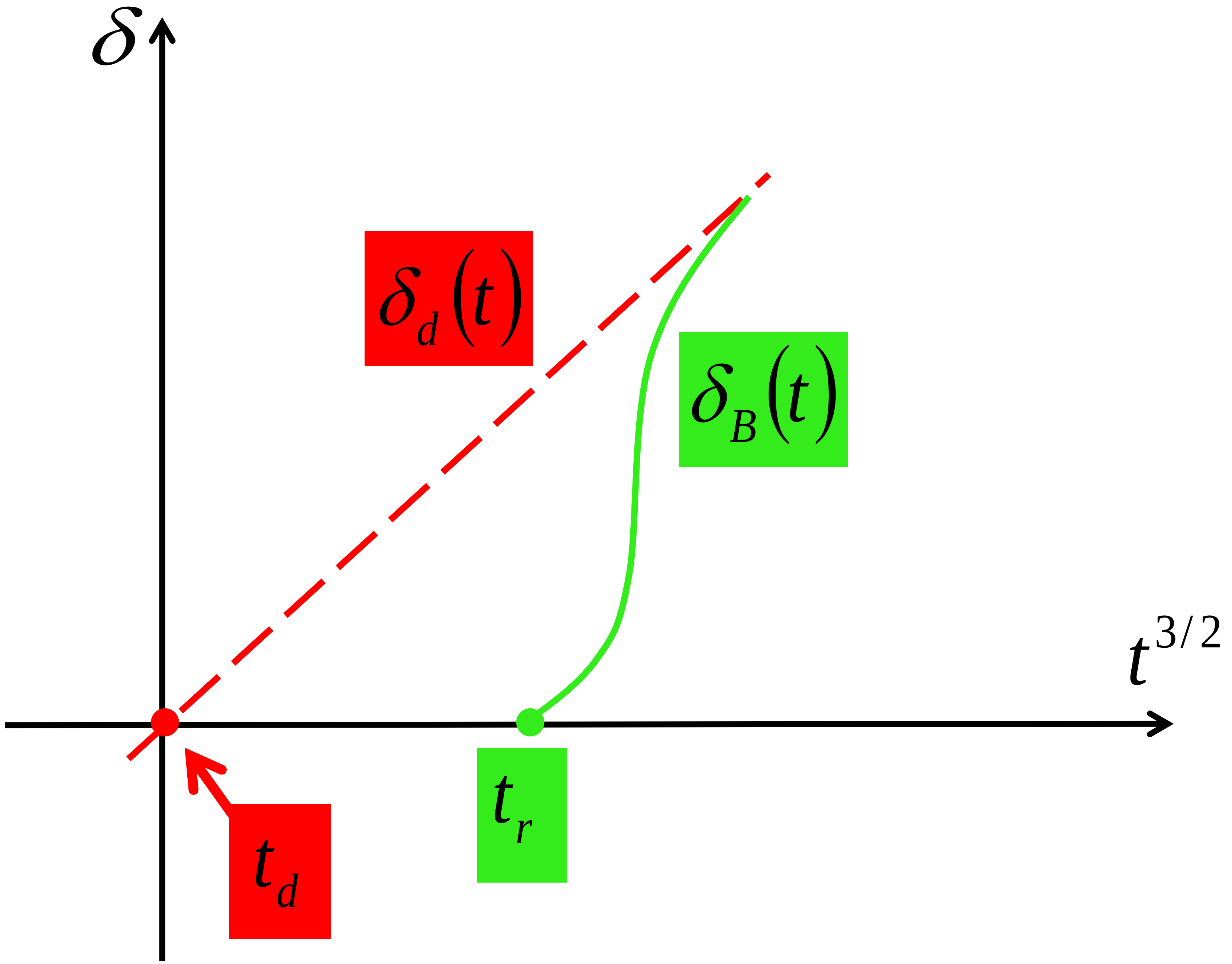
$$\frac{d}{dt}\left(t^{4/3}\frac{d\delta_b}{dt}\right)=\frac{2\delta_{DM}}{3t_d^{2/3}}$$

If we imply that

$$\delta_B(t_r)=0 \quad \text{and} \quad \dot{\delta}_B(t_r)=0$$

We obtain the solution

$$\delta_b(t)=\delta_{DM}\left(\frac{t}{t_d}\right)^{2/3}\left[1-3\left(\frac{t_r}{t}\right)^{2/3}+2\frac{t_r}{t}\right]$$



δ
δd(t)
δB(t)
t3/2
td
tr

Dark Matter.
Basic Properties of Dark Matter.

- *Dark matter must be* ***dark,*** *in the sense that it must have no (or extremely weak) interaction with photons*
- *Self-interaction of dark matter should be small. If dark matter particles can self scattered several times in galactic halo during the life time of the Universe, it would suffer gravi-thermal catastrophe*
- *Interaction with baryons must be weak also*
- *Dark matter particles do not belong to Standard Model of Physical Interactions*

Dark Matter

Hot, Warm, and Cold Dark Matter, beyond hydrodynamical approximation

В расширяющейся Вселенной есть несколько характерных моментов времени, важных для описания эволюции возмущений. Первый момент – это момент времени, когда длина волны возмущения становится равной длине горизонта частиц t_p (иногда говорят, что волна входит под горизонт). Второй момент времени t_{eq}, когда осуществляется переход от радиационно-доминированной стадии к стадии расширения Вселенной доминированной пылью. Затем следует момент времени рекомбинации водорода t_r, Вселенная становится прозрачной для реликтового излучения, давление фотонного газа резко падает, начинается рост возмущений в барионной компоненте материи. Наконец, последний из таких моментов t_Λ – начало доминирования темной энергии, рост возмущений «замораживается». Момент времени t_p зависит от длины волны возмущения и параметра Хаббла, а остальные моменты времени определяются только фоновыми плотностями соответствующих типов материи. Моменты времени расположены в следующем порядке: $t_{eq} < t_r < t_\Lambda$, а t_p может иметь любую величину от планковского времени до возраста Вселенной. Нас, впрочем, будет интересовать вполне определенный интервал времени, который соответствует волнам интересным с точки зрения образования крупномасштабной структуры Вселенной.

Частицы **холодной темной материи** (**Cold Dark Matter, CDM**) отщепляются от первичной плазмы будучи уже нерелятивистскими, а частицы **теплой темной материи (Warm Dark Matter, WDM) и горячей темной материи (Hot Dark Matter, HDM)** отщепляются еще релятивистскими. Различие между **теплой и горячей** DM определяется в момент перехода от RD (радиационно-доминированной) стадии к стадии доминирования пыли (DD). Если на стадии перехода частицы DM являются **нерелятивистскими**, то такая темная материя называется **теплой**. Если они остаются **релятивистскими**, то такая темная материя называется **горячей**.

Возмущения плотности в среде из бесстолкновительных частиц, которые движутся с релятивистскими скоростями и взаимодействуют только гравитационно, подвержены *затуханию Ландау*. В англоязычной литературе используется термин *free streaming*. Физический механизм процесса – свободное перемещение частиц с выравниванием плотности. Характерный масштаб такого процесса есть произведение характерной скорости частиц на характерное время изменения фоновой плотности. Характерное время изменения фоновой плотности – хаббловское время. Возмущения на масштабах меньше этого замываются из-за затухания Ландау. Затухание возмущений плотности в темной материи приводит также к затуханию возмущений гравитационного потенциала.

Проанализируем вначале затухание Ландау на примере холодной темной материи (CDM).

Эффект затухания появляется только при рассмотрения отдельных частиц, для его анализа необходимо использовать кинетические уравнения. Кинетическое уравнение для функции распределения бесстолкновительных частиц по координатам и импульсам *f(x,p)* имеет вид

$$p^{\alpha}\frac{\partial f}{\partial x^{\alpha}}+\Gamma^{\beta}_{\mu\nu}p^{\mu}p_{\beta}\frac{\partial f}{\partial p_{\nu}}=0$$

Правая часть равна нулю, поскольку интеграл столкновений равен нулю. Левая часть представляет собой релятивистское уравнение Лиувилля.

В случае нерелятивистских частиц и малых возмущений метрики на плоском фоне имеем уравнение:

$$\frac{\partial f}{\partial t}+\vec{v}\,\frac{\partial f}{\partial \vec{r}}-\frac{\partial \Phi}{\partial \vec{r}}\,\frac{\partial f}{\partial \vec{v}}=0$$

Причем, плотность темной материи определяется интегралом

$$\rho(t,\vec{r})=m\int d^{3}v f(\vec{v},t,\vec{r}),$$

а ϕ - гравитационный потенциал.

Введем величины, описывающие флуктуации. Функцию распределения представим в виде суммы: $f(x,p) = f^0(x,p) + \delta f(x,p)$. Поправку к потенциалу обозначим φ. Тогда кинетическое уравнение для флуктуаций функции распределения есть:

$$\frac{\partial \delta f}{\partial t} + \vec{v}\frac{\partial \delta f}{\partial \vec{r}} = \frac{\partial \varphi}{\partial \vec{r}}\frac{\partial f^0}{\partial \vec{v}}$$

Далее, также как в случае гидродинамического приближения, мы можем представить решение этого уравнения в виде интеграла Фурье и найти дисперсионное уравнение, которое покажет нам области неустойчивости и устойчивости.

Дисперсионное уравнение для волнового числа возмущения *k* и частот

$$k^2 = 4\pi Gm \int d^3v \frac{(\vec{k}\vec{v})}{\omega^2 - (\vec{k}\vec{v})^2} \left(\vec{k} \frac{\partial f^0}{\partial \vec{v}} \right)$$

В случае малых волновых чисел $k \to 0$

Дисперсионное уравнение есть $\omega^2 = -4\pi G\rho$

Частота является мнимой, вместо колебаний появляется декремент, описывающий экспоненциальное затухание и экспоненциальный рост. Так что, бесстолкновительные частицы в пределе малых *k* ведут себя как в гидродинамическом приближении.

Величину волнового числа, начиная с которого рост прекращается можно найти из уравнения дисперсии полагая $\omega=0$.

$$k_L^2 = 4\pi Gm \int d^3v \, \frac{f^0}{v^2}$$

или

$$k_L^2 = 4\pi G\rho \left\langle \frac{1}{v^2} \right\rangle$$

где

$$\left\langle \frac{1}{v^2} \right\rangle = \frac{\int d^3v \, \frac{f^0}{v^2}}{\int d^3v f^0}$$

Рассмотрим поведение частиц холодной темной материи, которые имеют массу m и «отщепляются» от равновесной плазмы при температуре T_d. Поскольку в момент отщепления частицы уже были нерелятивистскими, то их распределение описывается функцией Больцмана с некоторой эффективной температурой T_e.

Быстрый рост возмущений в темной материи начинается после момента t_{eq}, перехода от радиационно-доминированной к стадии к стадии доминирования пыли. Поэтому вычислим значение длины волны Ландау в этот момент времени.

$$\lambda_L = \frac{2\pi}{H(t_{eq})}\sqrt{\frac{T_e(t_{eq})}{m}}$$

Современное значение этой длины есть

$$\lambda_0 = (1+z_{eq})H_{eq}^{-1}\sqrt{\frac{T_{eq}^2}{mT_d}} \approx \sqrt{\frac{1\,GeV^2}{mT_d}} \quad pc$$

Рассмотрим частицы CDM с характеристиками m=100 ГэВ T_d ~ 10 МэВ, в этом случае масштаб длины Ландау составляет примерно 1 пк . Такие масштабы «замыты» затуханием Силка в ранней Вселенной, они нам неинтересны.

Рассмотрим теперь случай, когда темная материя является теплой. Пусть частицы являются фермионами. Функция распределения есть:

$$f^0(t) = f_0 \frac{1}{\exp\left\{\dfrac{mv}{T_e(t)}\right\}+1}$$

Соответственно средний квадрат скорости в минус первой степени есть

$$\left\langle \frac{1}{v^2} \right\rangle = 0.4\left(\frac{m}{T_e}\right)^2$$

Зависимость v^{-2} от температуры и массы другая, чем в случае CDM. Современное значение критической длины волны есть:

$$\lambda_L = 220\, kpc \left(\frac{100}{g(T_d)} \right)^{1/3} \left(\frac{1\, keV}{m} \right)$$

Здесь *g(T)* число степеней свободы при соответствующей температуре. Если *m* = 1 кэВ, то

$$\lambda_L = 220\, kpc \left(\frac{100}{g(T_d)} \right)^{1/3} \left(\frac{1\, keV}{m} \right) \approx 3\, Mpc$$

Очень важный вывод для космологии следует из физики нейтрино. Нейтрино – единственные из известных частиц, которые могут быть «горячей темной материей». Нейтринные осцилляции свидетельствуют также, что одно из нейтрино имеет массу. В случае, если масса порядка долей электронвольта, соответствующий масштаб есть:

$$\lambda_L \propto 200 \text{ Mpc}$$

Следует также добавить, что эти нейтрино не могут составлять значительную долю темной материи, а лишь небольшую её часть.

END

- *Dark matter is classified also in terms "hot" (Hot Dark Matter or **HDM)** and "cold" (Cold Dark Matter or **CDM**)*
- *Self-interaction of dark matter should be small. If dark matter particles can self scattered several times in galactic halo during the life time of the Universe, it would suffer gravi-thermal catastrophe*
- *Interaction with baryons must be weak also*
- *Dark matter particles do not belong to Standard Model of Physical Interactions*

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The key feature to these terms is the energy level of DM particles when they decouple from primordial plasma. If the DM particles were relativistic at the moment of decoupling the average impulse is equal to

$$p = \frac{E}{c}$$

Later on velocity of DM particles became nonrelativistic due to velocity reducing because of the Universe expansion, but the distribution function of energy-impulse will be the same.

Very important effect bends the matter power spectrum depending on the nature of DM. It take place when the mode of perturbation enters the horizon. It means that the wavelength of perturbation became less then the particle horizon size. In the case of HDM the free-streaming movement will erase the small scale perturbations, until the particles became non relativistic.

Weakly interacting Massive Particles (WIMPs)

- *Dark matter must be **dark,** in the sense that it must have no (or extremely weak) interaction with photons*
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